EXTENSION OF THE "REYNOLDS ANALOGY" THROUGH AN INTEGRAL METHOD APPROACH

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ABSTRACT

The "Reynolds Analogy" as well known, is a remarkable relationship, found by Reynolds, between the momentum and the energy transfer within the boundary layer in the case of an incompressible low speed flow over an isothermal flat plate, that allows, under the restriction of flows with uniform external boundary conditions, the strightforward evaluation of the heat flux at the wall in terms of the shear stress evaluated at the wall.

An extension of such analogy is proposed in the present paper for the cases in which the external flow boundary conditions are non uniform due to the establishment of accelerated flows along bidimensional and axisymmetric bodies; such cases have a wider interest for the pratical applications in the supersonic vehicle design.

The more general relationship between the heat flux at the wall, expressed in its adimensional form by the Stanton number, and the shear stress at the wall, expressed in adimensional form by the shear coefficient, is established on the basis of the momentum and energy boundary layer integral equations.

The analogy factor in the present formulation is expressed in terms of functions relating boundary layer quantities depending on a velocity gradient parameter, that can be evaluated on the basis of known results of similar solutions suitably correlated; the proposed approach evidentiates also the influence of the non similarity effects related to the variation along the body surface of the velocity gradient parameter and provides a mean to partially account for their influence on the analogy factor.

Simple formulas have been developed correlating the similarity flow results of Dewey and Gross for the case of a Prandtl number equal to 0,7, a power low viscosity-temperature relationship with $\omega = 0.7$ and a unit value of the hypersonic parameter appearing in the dissipation term of the energy equation characteristic of the supersonic flow conditions; these formulas provide in terms of a velocity gradient and a wall temperature parameter the values of the analogy factor on the basis of which the Stanton number and therefore the heat flux at the wall can be evalutated once the skin friction coefficient distribution along an assigned body surface is known.

INTRODUCTION

The advent of pratical applications of the high speed flows analysis associated with supersonic aircraft or space vehicles construction, has posed to the design engineer a series of challanging problems among which outstanding is the one of "Aerodynamic Heating" of simple and complex body shapes.

A severe heat transfer from the fluid to the body surface occurs under the high speed flow conditions as a result of the conversion the kinetic energy of motion by means of friction within the thin layer of the retarded flow that envelopes the body; such a heat transfer that may provoke inadmissible high temperatures of the body surface if not shielded needs to be accurately predicted.

The correct evaluation of the heat flux distribution along the surface of a body requires the detailed knowledge of the flow field that is established around the body itself; being impractical, for the high difficulties associated, the direct solution of the complete Navier-Stockes equations ruling as known the motion of viscous fluids, special solutions are usually obtained with the aid of the "Boundary Layer Theory"; such a theory provides, for engineering applications, a very pawerfull method for predicting, with high accurancy, the skin friction and the heat flux distributions at the surface of bodies that are the fundamental quantities needed in the design of an aerospace vehicle.

The basic idea of the "Boundary Layer Theory" due to Prandtl is well known: the flow field established around a body is subdivisible into two regions, an external one in which the fluid viscosity effects are neglected and an internal one where the fluid viscosity has on the contrary a predominant influence, while other terms of the complete equations can be neglected thus obtaining a simpler set of equations, the so called Prandtl or Boundary Layer equations.

Nevertheless even for simple shapes the mathematical difficulties for resolving these simplified equations are so severe that only by means of numerical procedures the complete solution providing the details of the flow field close to the body can be found.

The design engineer interest is confined in many cases only on the effects that the high speed flow produces on the body surfaces, that is to say on the skin friction and on the heat fluxes distribution; of pratical interest are therefore the "Integral Methods" that have been established to evaluate such properties without requiring the detailed knowledge of all the flow field established around the body.

One of the major difficulties in solving high speed flow boundary layer equations is the coupling between the dynamic and the thermal fields; at high speeds indeed the heat due to friction and compression within the boundary layer needs to be taken into account in the energy balance and therefore in addition to the usual dependence of the temperature field from the velocity field existing also at low speed conditions, the dependence of the velocity on the temperature field has to be accounted for, resulting in a complex coupling. Such a situation requires the combined solution of the dynamic and thermal fields, that is to say the momentum and energy conservation equations need to be solved contemporaneously to obtain the shear stress and the heat flux values at the body surface.

The classical low speed boundary layer equations allow on the contrary the separate treatment of the dynamic field on the basis of the momentum conservation equation alone, the temperature field being solved subsequently on the basis of the velocity field knowledge; as a consequence the heat flux is more simply evaluated on the basis of the skin friction value in virtue of the "Reynolds Analogy", when applicable.

Purpose of this note is to present an extension of the "Reynolds Analogy" derived on the basis of the integral momentum and energy equations that is applicable to the high speed flow conditions, also in presence of a non uniform external flow field; the classical "Reynolds Analogy" and the basic boundary layer equations in their local and integral form, utilized for obtaining the new relationship that allows the heat flux evaluation on the basis of the skin friction knowledge, are briefly recalled.

THE REYNOLDS ANALOGY

A remarkable relation of proportionality between the skin friction and the heat transfer was discovered in 1874 by Reynolds [1], which is known as "Reynolds Analogy between Momentum and Energy Transfer". This relation can easily be demonstrated on the basis of the classical boundary layer equations for the particularly simple case of low speed incompressible flow over an isothermal flat plate, that is to say in absence of pressure gradients and of frictional heat.

It turns out in fact that under such conditions the momentum equation (1) is identical to the energy equation (2) on condition that the Prandtl number $P_r = \mu c_p / K$ is equal to unity; the boundary conditions being as well identical at the wall (3) and at the edge of the boundary layer (4).

$$\rho\left(u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y}\right) = \mu \frac{\partial^2 u}{\partial y^2} \tag{1}$$

$$\rho c_p \left(u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y}\right) = K \frac{\partial^2 T}{\partial y^2} \tag{2}$$

$$u = 0 \qquad T - T_w = 0 \qquad \text{at } y = 0 \tag{3}$$

$$u = u_e \qquad T - T_w = T_e - T_w \qquad \text{at } y = \delta \tag{4}$$

The solution for u/u_e and $(T-T_w)/(T_e-T_w)$ have therefore the same identical form, the velocity and temperature profiles are similar and the dynamic boundary layer has the same thickness of the thermal one.

Being the local skin friction and the heat flux at the wall respectively:

$$\tau_w \equiv \mu (\partial u/\partial y)_0 = \mu u_e \left[\frac{\partial}{\partial y}(u/u_e)\right]_0 \tag{5}$$

$$q_w \equiv K (\partial T/\partial y)_0 = K (T_e - T_w)\left[\frac{\partial}{\partial y}\left(\frac{T-T_w}{T_e-T_w}\right)\right]_0 \tag{6}$$

it follows from the identity of the veloctiy and temperature fields, being equal the gradients at the wall, that the heat flux can be expressed as :

$$q_w = \frac{K}{\mu}\left[(T_e - T_w)/u_e\right] \cdot \tau_w \tag{7}$$

that is to say the heat flux is proportional to the skin friction.

Introducing the skin friction coefficient

$$C_f = \tau_w / (1/2\, \rho_e u_e^2) \tag{8}$$

and the Stanton number

$$St \equiv (Nu / Re\, Pr) = q_w / \rho_e c_p u_e (T_e - T_w) \tag{9}$$

the above recalled "Reynolds Analogy" takes the very simple classical form:

$$St = 1/2\; C_f \tag{10}$$

Extensive use has been made of such a simple relation, suitably modified to broaden its range of validity, to obtain, from known skin friction distributions, evaluated according to existing methods based on momentum equation solutions, the distribution of the heat flux without solving the energy equation contemporaneously.

In its original form the "Reynolds Analogy" is affected as recalled by severe limitations: Prandtl number equal to one, no account for frictional heating, incompressible flow, absence of gradients in the external flow at the edge of the boundary layer.

The first extension has been introduced to account for a Prandtl number different from unity, it has been found that the Stanton number remains even for $P_r \neq 1$ proportional to the coefficient of skin friction,excepts that the factor of proportionality known as Reynolds Analogy factor is a function of the Prandtl number. Several forms of the analogy factor have been proposed, the more widely accepted one is due to Colburn

$$S = (P_r)^{-2/3} \tag{11}$$

Sucessively the frictional heat has been taken into account for incompressible flows, by Eckert and Weise [3] who found that the equations remain unchanged if the Stanton number is defined with reference to the temperature difference $(T_{aw} - T_w)$ instead than $(T_e - T_w)$ being T_{aw} the adiabatic wall temperature.

For the case of compressible laminar flow over an isothermal flat plate, the equation relating the temperature field to the velocity one remain approximatively the same as for incompressible flows and for $P_r = 1$ even exactly the same; the analogy therefore can be extended on condition that the Stanton number is formed with the temperature difference $(T_{aw} - T_w)$ as previously discussed.

The importance of the Reynolds analogy, demonstrated for the laminar flow condition, is even more increased by the consideration that it retains its validity under the conditions previously discussed, also in the case of turbulent flows.

The major limitation of the Reynolds Analogy, that written in its more general form to account for the Prandtl number effects, turns out to be

$$St = S \cdot 1/2\, C_f \qquad (12)$$

with S the analogy factor equal to $P_r^{-2/3}$, is the applicability only to uniform flows in absence of gradients at the edge of the boundary layer.

Exact boundary layer equations solutions have shown, as we will recal in detail, that when the velocity or pressure gradient parameter is different from zero, remarkable deviations from the classical analogy exist; accelerated flows, that is to say flows with positive velocity gradients, lead to an analogy factor that, when account is taken of the Prandtl number effect, is smaller than one. The influence of the wall temperature, kept constant along the body in the exact solutions, is as well quite noticeable; the analogy factor being lower the higher the wall temperature.

In spite of this evidence in too many cases the Reynolds Analogy has been nevertheless used to obtain from the skin friction values the heat flux distribution along bodies, also in presence of velocity gradients at the edge of the boundary layer.

An attempt of extension of the Reynolds Analogy will be presented in this note for compressible laminar boundary layer flows, properly accounting for the effects produced by the non uniformity of the external flow field; in particular the condition of accelerated flows that has an high pratical interest for its engineering applications to aircraft and missile design is treated.

THE BOUNDARY LAYER EQUATIONS IN COMPRESSIBLE FLOW

The well known equations for the laminar compressible boundary layer flows over two dimensional and axisymmetric bodies are recalled in their classical forms: local equations valid at any point within the boundary layer and integral or global equations valid as a mean value across the boundary layer thickness.

Local Equations

Using the (x, y) orthogonal coordinate system, with x measured along the body surface (origin at the nose or leading edge) and y measured along the outwards normal at the body surface, the local boundary layer equations expressing the conservation of mass, momentum and energy within the boundary layer, for the steady flow of a perfect gas over an unyawed body are:

Continuity

$$\frac{\partial}{\partial x}\left(\rho u r^j\right) + \frac{\partial}{\partial y}\left(\rho v r^j\right) = 0 \qquad (13)$$

X Momentum

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \frac{1}{r^j}\frac{\partial}{\partial y}\left(\mu r^j \frac{\partial u}{\partial y}\right) \qquad (14)$$

Y Momentum

$$\frac{\partial p}{\partial y} = 0 \qquad (15)$$

Energy

$$\rho\left(u\frac{\partial H}{\partial x} + v\frac{\partial H}{\partial y}\right) = \frac{1}{r^j}\left[\frac{\partial}{\partial y}\left(\frac{\mu}{P_r} r^j \frac{\partial H}{\partial y}\right) + \frac{\partial}{\partial y}\left(\frac{\mu}{P_r}\left(P_r - 1\right) r^j \frac{\partial^2 u}{\partial y^2}\right)\right] \qquad (16)$$

with u, v the velocity components along the x and y directions, H the total enthalpy $H = h + 1/2\, u^2$ and r = r (x, y) the local distance measured from the axis of symmetry.

The above system of non linear partial differential equations is obtained under the assumption that the boundary layer thickness is small compared to the longitudinal body radius of curvature and that the centrifugal forces are negligible: they contain the transverse curvature terms specified by r^j according to the formulation by Yasuhara [4] for the two dimensional flows $j = 0$, for the axisymmetric flows $j = 1$.

The perfect gas assumptions allow to establish the remaining equations enabling the resolution of the system of conservation equations

– Equation of state $\quad p = \rho R T \qquad (17)$

– Enthalpy relation $\quad h = c_p T \qquad (18)$

Associated to the system differential conservation equations, valid at every point within the boundary layer, are the boundary conditions established at the wall (y = 0) and at the outer edge of the boundary layer ($y = \delta$) where the external inviscid flow conditions are matched; these conditions uniquely determine the solution.

At the wall the requirement of no slip imposes the velocity condition:

$$u = v = 0 \qquad at \quad y = 0 \qquad (19)$$

while the temperature may satisfy the condition of no heat transfer (adiabatic wall) or of specified value

$$\left(\frac{\partial T}{\partial y}\right)_0 = 0 \quad or \quad (T)_0 = T_w \quad at \quad y = 0 \qquad (20)$$

At the outer boundary, the edge of the boundary layer, the values of the velocity and temperature are specified by the inviscid flow solution

$$u = u_e \quad and \quad T = T_e \quad at \quad y = \delta \qquad (21)$$

In spite of the boundary layer assumptions, that introduce remarkable simplifications with respect to the complete Navier-Stokes equations, the mathematical difficulties associated with the solution of the boundary layer equations are relevant expecially when the compressibility, the pressure gradient and the heat transfer effects are accounted for; only very few general solutions have therefore been obtained, that are valid under restrictive conditions, an important class of such solutions are the ones known as "Similar Solutions".

"Similar Solution"

Coordinate transformations have been studied starting from Blasius that allow the reduction of the boundary layer partial differential equations to ordinary differential equations thus facilitating their solution.

Several transformations exist, the more general is the one of Lees-Dorodnitsyn that contains the Blasius, the Mangler and the Levy ones

$$\xi(x) = \int_0^x C\,(\rho_e \mu_e)\, r_w^{2j}\, dx \qquad (22)$$

$$\eta(x,y) = u_e / \sqrt{2\xi} \cdot \int_0^y \rho\, r^j\, dy \qquad (23)$$

By applying such transformation to the local equation for the mass (13), momentum (14) and energy (16) conservations, and by introducing as dependent variables

$$f_\eta = \partial f / \partial \eta = u/u_e \qquad (24)$$

$$g = H/H_e \qquad (25)$$

the momentum and energy equations can be transformed, with the aid of the continuity equation, into the classical form:

$$(\lambda R f_{\eta\eta})_\eta + f f_{\eta\eta} = (2\xi/u_e) \frac{du_e}{d\xi} \left(f_\eta^2 - \rho_e/\rho \right) \qquad (26)$$

$$(\lambda R g_\eta)_\eta + f g_\eta = \frac{u_e^2}{2H_e} \left(2\lambda R (1/P_r - 1) f_\eta f_{\eta\eta} \right)_\eta \qquad (27)$$

with $R = (r/r_w)^{2j}$ and $\lambda = 1/C\,(\rho\mu/\rho_e\mu_e)$ being $C = \rho_w\mu_w/\rho_e\mu_e$ the Chapman Rubesin constant.

All derivatives with respect to ξ within the boundary layer have been neglected, thus reducing the partial differential equations in η and ξ to ordinary differential equations in the single variable η.

Associated to the above equations are the boundary conditions.

At the surface of the body

$$f_\eta(0) = 0 \qquad g(0) = g_w \qquad (28)$$

At the edge of the boundary layer

$$f_\eta(\infty) = 1 \qquad g(\infty) = 1 \qquad (29)$$

In order that the transformed equations (26) and (27) be similar, that is to say in order to have solutions independent on the variable ξ, it is necessary that the dependent variables are functions only of η; restrictive conditions on the free stream pressure and temperature distribution at the outher edge of the boundary layer have to be imposed to satisfy the similarity requirements.

The momentum and energy equations reduce for similar flows to the form

$$(\lambda f_{\eta\eta})_\eta + f f_{\eta\eta} = \beta \left[f_\eta^2 - (t_w \lambda)^{\frac{1}{\omega} - 1} \right] \qquad (30)$$

$$(\lambda g_\eta / P_r)_\eta + f g_\eta = \left[(1/P_r - 1) \frac{2\sigma\lambda}{(1 - t_w)} f_{\eta\eta} f_\eta \right]_\eta \qquad (31)$$

with β the modified Falkner-Skan parameter, related to the velocity gradient, defined as $\beta = 2m = (2\xi/u_e)(T_0/T_e) \frac{du_e}{d\xi}$ and $\lambda = \rho\mu/\rho_w\mu_w = \lambda(\eta)$ independent on ξ and $t_w = T_w/T_0$ the normalized wall temperature parameter with T_0 the free stream stagnation temperature, and $\sigma = (u_\infty^2/2H_e)(u_e/u_\infty)^2$ the hypersonic parameter and ω, the exponent of the viscosity - temperature relation.

Solutions to the above similar equations with appropriate boundary conditions have been obtained among others by Dewey and Gross [5] as a function of the pressure gradient parameter β with, t_w, σ, ω, P_r, as parameters.

Such solutions are called similar because the velocity and the temperature or enthalpy profiles are "affine" or "similar" at various locations along the body, differing only by a scale factor.

In cases when the simularity conditions are not applicable no general method of solution has yet been found and use has to be done of numerical solutions.

A very relevant approach that enlarges the importance of similar solutions and allows the approximate treatment of non similar cases is the one discussed by Lees [6] known as the "Local Similarity" approach.

All the therms in the boundary layer similar equations dependent on ξ arising from the external flow or wall conditions, are assumed to have their local values that can change arbitrarely along the body surface.

"Local Similarity" represents essentially a patching together of local similar solutions; the x-wise story of the flow is ignored except as it is contained in the external and wall conditions; the validity of such approximation depends on the fact that the external flow properties vary slowly with respect to ξ that is to say along the x coordinate.

Integral or global equations

Among the approximate methods very powerful have turned out to be the integral ones based essentially on the Von Karman integral approach idea.

The intregral methods treating the boundary layer problem from a global point of view give evidence only to these quantities that representing the boundary layer effect as a whole directly allow the definition of the shear stress and the heat flux at the wall, such quantities are obtained not on the basis of derivatives of the velocity and enthalpy profiles but in relation to the evolution along the body surface of some appropriate integral quantities defined as boundary layer dynamic and thermal thicknesses.

The integral equations can be established by making the balance of mass, momentum and energy for a finite volume of gas encompassing the complete boundary layer thickness, instead than for the elementary volume, or formally can be obtained integrating with respect to "y" the local boundary layer equations from the wall up to the external edge of the boundary layer. A set of ordinary differential equations in terms of the boundary layer thickness integral parameters are obtained, that satisfy the boundary conditions at the wall and at the outer edge of the boundary layer.

The original local boundary layer equations are not satisfied for every fluid particle within the boundary layer, but exactly only in a stratum near to the wall and near to the region of transition to the external flow, while in the boundary layer core region the differential equations are satisfied only as an average.

The classical integral technique has been originally applied by Von Karman [7] to the momentum equation of an incompressible boundary layer flow; the method has been extended to the high speed compressible boundary layer conditions by introduction of an energy integral equation. The simultaneous solution of the two integral equations provides the approximate solution of the boundary layer global properties at the wall.

The frontier of the boundary layer, up to which the integration has to be made, is fixed at the distance $y = \delta(x)$ from the wall at which the grandients in the velocity and temperature profile due to the wall presence are vanishing and the external inviscid flow conditions are matched; the values of the velocity components, the density, the temperature and the enthalpy there are respectively:

$$u_e,\ v_e,\ \rho_e,\ T_e,\ H_e \qquad \text{at } y = \delta \tag{33}$$

Three fundamental boundary layer thicknesses are usually defined representing respectively the loss of mass flow, of momentum, and of energy associated to the presence of the boundary layer with respect to the external flow values.

Displacement thickness

$$\Delta = \int_0^\delta \left(1 - \rho u/\rho_e u_e\right) dy \tag{34}$$

Momentum thickness

$$\Theta = \int_0^\delta \rho u/\rho_e u_e \left(1 - u/u_e\right) dy \tag{35}$$

Energy or totale enthalpy thickness

$$\Lambda = \int_0^\delta \rho u/\rho_e u_e \left(1 - H/H_e\right) dy \tag{36}$$

that represent the integral parameters in term of which the integral boundary layer equations are expressed.

By integrating from the wall ($y = 0$) to the boundary layer edge ($y = \delta$) the continuity, (13) X momentum (14) and the energy (16) local equations the corresponding global or integral equations are obtained.

$$\frac{d\Delta}{dx} + \Delta(1-B)\left[\frac{1}{u_e}\frac{du_e}{dx} + \frac{1}{\rho_e}\frac{d\rho_e}{dx} + \frac{1}{r^j}\frac{dr^j}{dx}\right] = v_e/u_e \tag{37}$$

$$\frac{d\theta}{dx} + \theta\left[(F+2)\frac{1}{u_e}\frac{du_e}{dx} + \frac{1}{\rho_e}\frac{d\rho_e}{dx} + \frac{1}{r^j}\frac{dr^j}{dx}\right] = \tau_w/\rho_e u_e^2 \tag{38}$$

$$\frac{d\Lambda}{dx} + \Lambda\left[\frac{1}{u_e}\frac{du_e}{dx} + \frac{1}{\rho_e}\frac{d\rho_e}{dx} + \frac{1}{r^j}\frac{dr^j}{dx}\right] = q_w/\rho_e u_e H_e \tag{39}$$

with $B = \delta/\Delta$ and $F = \Delta/\theta$ called shape parameters being dependent uniquely on the skape of the velocity and entlphy profiles within the boundary layer.

The continuity integral equations relates the displâcement thickness Δ to the deviation of the external flow field velocity vector v_e/u_e induced by the presence of the body and due to the boundary layer building up; it involves also the boundary layer thickness δ (via the shape parameter B) in addition to the external flow and body configuration parameters.

The momentum integral equation relates the momentum thickness Θ to the shear stress at the wall expressed as Fanning friction factor $f = \tau_w/\rho_e u_e^2$ it involves also the displacement thickness Δ (via the shape parameter F) in addition to the external flow and body configuration parameters.

The energy integral equation relates the energy thickness Λ directly to the heat flux at the wall $\dot{q}_w$ expressed in the form of Stanton number $St = q_w/\rho_e u_e H_e$ it involves only the external flow and the body configuration parameters.

The classical approach to solve with the aid of the integral equations, the boundary layer problems is to assume suitable expression for the velocity distribution $u/u_e = f(y/\delta)$ and for the temperature distribution $(T-T_w)/(T_e-T_w) = g(y/\delta)$ satisfying the appropriate boundary conditions at the wall and at the outer edge, containing as free parameter a suitably chosen boundary layer thickness, and to express all the terms as a function of this free parameter, that is determined with the aid of the integral equation.

Pohlhausen [8] has developed a method successively ameliorated by Holstein and Bohlen [9], on the basis of the Von Karman momentum equation for two dimensional incompressible flow that is based on a fourth degree approximation of the velocity profile; the method has been extended by Tomotika [10] to include axisymmetric bodies of revolution.

The same approach has been adopted by Gruschwitz [11] for solving compressible boundary layers flows along adiabatic wall accounting for the momentum and Kinetic energy integral equations, while Morris and Smith [12] developed a very general method for heat conducting walls that includes the case of variable surface temperature. Another generalisation of the Pohlhausen method is due to Kalikhman [13] who solved the momentum and energy integral equation assuming fourth order velocity and temperature profiles.

In general some difficulties arise in the practical applications of such method leading to the solution of ordinary differential equations so that only seldom they have been applied to specific design conditions.

Another approach is based on the concept of combination of the integral method and similar solutions that was first introduced by Thwaites [14] for the incompressible flows with arbitrary pressure gradients, he pointed out that to calculate the boundary layer thickness and the skin friction distribution it is not necessary to introduce explicitly assumptions concerning the velocity profile, as all the methods derived from Pohlhausen do; it is enough to obtain functional relations between the shear stress at the wall, the shape parameter F and a local pressure gradient parameter $(\theta^2/\nu)\, du_e/dx$ and solve the ordinary differented equation derived from the momentum integral equation.

To obtain such relations Thwaites used the known similar solution for incompressible-boundary layers obtaining a simple solution by quadrature of the differential equation.

The approach has been extended by Rott and Crabtree [15] to compressible flow over an adiabatic body, while Cohen and Reshotko [16] have treated the case of bodies with heat transfer.

In case of bodies with heat transfer the momentum integral and the energy integral equations cannot be satisfied simultaneously so that Cohen and Reshotko ignored the energy equation and obtained the heat flux from the similar solutions through the correlation determined only on the basis of the momentum integral equations alone; if the energy integral equation is used however in general two different answers are obtained for the heat flux.

The problem of utilizing existing similar solutions in a momentum integral method has been readdressed by Hayes [17] through a modification of the classical approach by expressing displacement, momentum and energy defect thicknesses in a dimensionless from directly calculable from similar solution.

$$\Delta^* = \int_0^\infty (g - f_\eta)\, d\eta$$

$$\Theta^* = \int_0^\infty f_\eta (1 - f_\eta)\, d\eta$$

$$\Lambda^* = \int_0^\infty f_\eta \,(1 - g)\, d\eta$$

Transformed momentum integral and energy integral equations have been obtained taking into account the relations existing in case of similar solution and an inconsistency has been put into evidence on the evaluation of the heat flux distribution at the wall from the energy equation, depending on whether the complete equation or its similar form is used; such differences disappear only if, Θ^* is proportional to Λ^* along the body surface.

To avoid such a problem Hayes developed a method for solving simultaneously the momentum and energy integral equations that gives the skin friction and the heat transfer at the wall in a self consistent form; the solution of the set of equations is not strightforward and may pose difficulties of mathematical nature particularly for what concerns the stability, being required a numerical solution.

A very attractive alternative method of solution of the integral boundary layer equations has been established by Michel [18], originally on the basis of the momentum equation alone, allowing the determination of the evolution along the surface of two dimensional and axisymmetric bodies, of the momentum thickness and consequently of the shear stress at the wall.

The shear stress coefficient and the shape parameter F are expressed as function of the momentum thickness Θ under the assumption that the flat plate solution dependence is not altered by the presence of the pressure gradients. The integral momentum equation is therefore reduced to an ordinary differential equation in terms only of the momentum thickness that is determined once known the external flow field velocity or pressure distribution and the body configuration variation; in such a way the gradients effect are accounted for at least in their primary effects.

On the basis of the momentum thickness distribution the shear stress at the wall is evaluated reutilizing the same flat plate relations used in the resolution; the heat flux distribution along the body surface is determined without resorting to the energy integral equation, but simply by applying the classical Reynolds analogy in its flat plate form, also in presence of velocity or pressure gradient in the external flow field.

The same technique has been followed successively by Michel himself [19] to directly determine the heat flux at the wall of a two dimensional or axisymmetric body in presence of pressure on velocity gradients in the external flow field on the basis of only the energy integral equation.

The heat flux at the wall has been related to the local value of the energy thickness Λ, also in such a case the flat plate solution has been utilized; to relate the heat flux to the energy thickness use has been made of the Reynolds analogy factor determined in the flat plate condition, so that the validity of such analogy has been assumed implicitly also in presence of gradients in the external flow properties.

The energy integral equation is therefore reducted to a sinple ordinary differential equation in terms only of the energy thickness Λ that is determined once the external pressure or velocity distribution and the body shape are known; also in such a case the gradients effect not accounted in the relation between the flux and the energy thickness, are accounted for in the determination of the energy thickness at last for their primary effects.

The Michel formulations are simple and the calculations are very strighforward so that they have been used in several applications even if they rely on the assumption that the external flow gradients do not affect the relation between shear stress coefficients, shape parameters and momentum thickness, that are assumed to be extendable as well as the Reynolds analogy from the flat plate condition.

PRESENT APPROACH TO INTEGRAL EQUATIONS SOLUTION

In order to improve the accuracy in the computation of heat transfer in compressible boundary layer flows and to avoid inconsistencies in the results, it is evident from what previously recalled that the energy integral equation must also be considered in addition to the classical momentum integral equation.

In the present approach both the momentum integral equation

$$\tau_w / \rho_e u_e^2 = \frac{d\theta}{dx} + \theta\left[(F+2)\frac{1}{u_e}\frac{du_e}{dx} + \frac{1}{\rho_e}\frac{d\rho_e}{dx} + \frac{1}{r^i}\frac{dr^i}{dx}\right] \tag{40}$$

and the energy integral equations

$$q_w / \rho_e u_e H_e = \frac{d\Lambda}{dx} + \Lambda\left[\frac{1}{u_e}\frac{du_e}{dx} + \frac{1}{\rho_e}\frac{d\rho_e}{dx} + \frac{1}{r^i}\frac{dr^i}{dx}\right] \tag{41}$$

are simultaneously dealt.

A new shape parameter, defined as the ratio of the energy defect thickness Λ to the momentum defect thickness θ

$$D = \Lambda / \theta \tag{42}$$

depending from the shape both of the velocity and the enthalpy profiles within the boundary layer, is introduced.

The energy integral equation is rewritten in terms of this new shape parameter

$$q_w / \rho_e u_e H_e = D \frac{d\theta}{dx} + \theta \frac{dD}{dx} + D\theta \left[\frac{1}{u_e}\frac{du_e}{dx} + \frac{1}{\rho_e}\frac{d\rho_e}{dx} + \frac{1}{r^j}\frac{dr^j}{dx} \right] \tag{43}$$

Combining it with the momentum integral equation a new form of the integral energy equation is obtained: that expresses the heat flux at the wall in terms of the shear stress at the wall, the momentum thickness θ, and in addition the shape parameters D and F, that is to say that involves the displacement, momentum and energy defect thickness.

$$q_w / \rho_e u_e H_e = D\, \tau_w / \rho_e u_e^2 + \theta \left\{ \frac{dD}{dx} - D(1+F) \frac{1}{u_e}\frac{du_e}{dx} \right\} \tag{44}$$

The influence of the variations of the external inviscid flow field boundary conditions is accounted for indirectly through the shear stress term that has to be evaluated taking properly into account the external flow non uniformities, and directly by the term involving the velocity gradient.

The above relation contains as well the influence of the variation of the shape parameters D along the surface of the body that is expected from the results of Hayes to have a strong influence on the heat flux at the wall.

The new energy integral equation (44) can be solved once the value of the shear stress at the wall is suitably determined from the resolution of the integral momentum equation (40).

Values of the shear stress term can be obtained according to various different methods, for uniformity of treatment we will outline a new approach consistent and similar to the one we are proposing for the energy integral equation.

Use of the similar solution results will be made in the present approach, as in other methods, to obtain the dipendence on the external flow field conditions of the quantities appearing into the momentum and energy equations that are expressed in terms of the parameters and functions defined in the treatment of the similar solutions

$$\tau_w / \rho_e u_e^2 = c \mu_e r^j f_{\eta\eta}(0) / \sqrt{2\xi} \tag{45}$$

$$\Delta = \left(\sqrt{2\xi} / \rho_e u_e r^j \right) \left(T_0 / T_e \right) \left[I_1 - T_e/T_0\, I_2 \right] \tag{46}$$

$$\theta = \left(\sqrt{2\xi} / \rho_e u_e r^j \right) \int_0^\infty f_\eta (1 - f_\eta)\, d\eta \tag{47}$$

$$\Lambda = \left(\sqrt{2\xi} / \rho_e u_e r^j \right) \int_0^\infty f_\eta (1 - g)\, d\eta \tag{48}$$

with

$$I_1 = \int_0^\infty (1 - f_\eta^2)\, d\eta - (1 - t_w) \int_0^\infty (1-g)/(1-g_w)\, d\eta \tag{49}$$

$$I_2 = \int_0^\infty f_\eta (1 - f_\eta)\, d\eta \tag{50}$$

defined in agreement with the analysis of Dewey and Gross [5].

The following form of the quantities appearing into the momentum and energy equations can accordingly be established

$$\theta = \frac{c \mu_e}{\rho_e u_e} \cdot f_{\eta\eta}(0)\, I_2 \left(\tau_w / \rho_e u_e^2 \right)^{-1} \tag{51}$$

$$\frac{\theta}{u_e} \cdot \frac{du_e}{dx} = \left(c\sqrt{2\xi} / \xi \right) \mu_e r_w^j \left(T_e / T_0 \right) m I_2 = \frac{2 m I_2 \left(\tau_w / \rho_e u_e^2 \right)}{f_{\eta\eta}(0) \left(T_0 / T_e \right)} \tag{52}$$

$$\theta \frac{d}{dx} = c\sqrt{2\xi}\, \mu_e r_w^j I_2 \frac{d}{d\xi} = \frac{2\xi I_2}{f_{\eta\eta}(0)} \frac{\tau_w}{\rho_e u_e^2} \frac{d}{d\xi} \tag{53}$$

$$\begin{aligned} \frac{d\theta}{dx} = & \left(\frac{c \mu_e}{\rho_e u_e} \right) \left(\tau_w / \rho_e u_e^2 \right)^{-1} \left\{ \frac{d\left(f_{\eta\eta}(0) \cdot I_2 \right)}{dx} + \right. \\ & \left(f_{\eta\eta}(0)\, I_2 \right) \left[\frac{1}{\mu_e}\frac{d\mu_e}{dx} - \frac{1}{u_e}\frac{du_e}{dx} - \frac{1}{\rho_e}\frac{d\rho_e}{dx} \right] + \\ & - \left(f_{\eta\eta}(0)\, I_2 \right) \left(\tau_w / \rho_e u_e^2 \right)^{-1} \frac{d}{dx} \left(\tau_w / \rho_e u_e^2 \right) \end{aligned} \tag{54}$$

The shape parameters, as well can be put into relationship with the similar solution functions, obtaining, from the Dewey and Gross [5] results

$$F \equiv \Delta / \theta = \left(T_0/T_e \cdot I_1 - I_2 \right) / I_2 \tag{55}$$

and, utilizing the results found by Hayes [17] for the energy integral equation for similar flows:

$$D \equiv \Lambda / \theta = g_\eta(0) / P_r I_2 \tag{56}$$

A set of three parameters are introduced for simplification of the treatment:

$$\Gamma \equiv I_2 / f_{\eta\eta}(0) = \Gamma \left(m, P_r, \sigma, t_w, \omega \right) \tag{57}$$

$$\Sigma \equiv \frac{1 + F}{T_0 / T_e} = I_1 / I_2 = \Sigma \left(m, P_r, \sigma, t_w, \omega \right) \tag{58}$$

$$\Psi \equiv f_{\eta\eta}(0)\, I_2 = \Psi \left(m, P_r, \sigma, t_w, \omega \right) \tag{59}$$

that can be easily evaluated on the basis of the similar solutions results.

To evaluate such parameters, as previously pointed out, use will be made of the results of Dewey and Gross [5] who have computed and tabulated for a large number of boundary layer solutions the values of the quantities

$$f_{\eta\eta}(0),\ g_\eta(0),\ I_1,\ I_2$$

as a function of the basic parameters :

m velocity parameter, P_r Prandtl number, t_w wall temperature, ω viscosity exponent and σ hypersonic parameter, previously defined.

Momentum integral equation

Taking into account for the quantities appearing into the momentum integral equation the form previously obtained and introducing the parameters above defined the equation can be rewritten as:

$$\frac{\tau_w}{\rho_e u_e^2} = \left(\tau_w/\rho_e u_e^2\right)^{-1}\left(\frac{C\mu_e}{\rho_e u_e}\right)\left\{\frac{d\psi}{dx} + \psi\left[1/\mu_e \frac{d\mu_e}{dx} - 1/u_e \frac{du_e}{dx} - 1/\rho_e \frac{d\rho_e}{dx}\right] + - \psi\left(\tau_w/\rho_e u_e^2\right)^{-1}\frac{d}{dx}\left(\tau_w/\rho_e u_e^2\right)\right\} + \left(\tau_w/\rho_e u_e^2\right)^{-1}\cdot \left(\frac{C\mu_e}{\rho_e u_e}\right)\psi\left\{(2+F)\frac{1}{u_e}\frac{du_e}{dx} + \frac{1}{\rho_e}\frac{d\rho_e}{dx} + \frac{i}{r}\frac{dr}{dx}\right\} \tag{60}$$

By introducing a skin friction parameter or Fanning factor defined as

$$f = \tau_w/\rho_e u_e^2 \tag{61}$$

and two functions depending on the local external inviscid flow boundary conditions and on the local boundary layer parameters, defined as

$$Q(x) = -R/\psi \tag{62}$$

$$P(x) = -\left(1/\psi \frac{d\psi}{dx} + T + (1+F)U\right) \tag{63}$$

with

$$R = \rho_e u_e / C\mu_e$$

$$T = \frac{1}{\mu_e}\frac{d\mu_e}{dx} + \frac{i}{r}\frac{dr}{dx}$$

$$U = \frac{1}{u_e}\frac{du_e}{dx}$$

three functions related only to the outer edge flow conditions and to the body shape, the momentum integral equations assumes the form:

$$\frac{df}{dx} - f^3 Q + f P = 0 \tag{64}$$

By recognizing that the resulting first order and first degree differential equation is of the Bernoulli type, the solution can be lead to the one of the linear differential equations through the transform of variable $Y = f^{-2}$ by setting $\bar{P} = -2P$ and $\bar{Q} = -2Q$; the solution therefore turns out to be simply

$$Y = \left[Y_0 + \int_0^x \bar{Q}\exp\left(\int_0^x \bar{P}dx\right)dx\right] \Big/ \exp\left(\int_0^x \bar{P}dx\right) \tag{65}$$

that is to say the skin friction coefficiente is given by the relationship

$$f = \frac{\mu_e u_e^{1+F} r^i \psi}{\left\{\left[\left(\mu_e u_e^{1+F} r^i \psi f^{-2}\right)_0\right]^2 + 2\int_0^x \frac{\rho_e \mu_e}{C} u_e^{2F+3} r^{2i}\psi\, dx\right\}^{1/2}} \tag{66}$$

According to the above formula, the value of the skin friction can be easily obtained in terms of known functions evaluated at the edge of the boundary layer, once the shape of the body is known, if the values of the function ψ and F are related to the inviscid flow conditions through the similar solutions; in deriving the above solution the value of the shape parameter F has been assumed to be slowly varying along the body surface so that the local value can be utilized.

Wall temperature effects are accounted for in the evaluation of f through values of ψ and F, and in addition by the Chapman Rubesin constant $C = \rho_w \mu_w / \rho_e \mu_e$.

Energy integral equation

Similarly, taking into account for the quantities appearing in the energy integral equation (eq 44) the form previously obtained (eq 51) (eq 52) (eq 53) and introducing the parameters defined, the equation can be rewritten as:

$$q_w/\rho_e u_e H_e = \tau_w/\rho_e u_e^2 \left\{D + 2\Gamma\left(\xi\frac{dD}{d\xi} - D\Sigma m\right)\right\} \tag{67}$$

The heat flux at the wall can be simply calculated, once the shear stress value at the wall is known, provided the values of the parameters D, Γ, Σ, dependent on the local velocity gradient through the parameter "m", are obtained in terms of external inviscid flow field by means of the similar solution results.

The energy integral equation shows as well through the term

$$X \equiv 2\Gamma\xi\frac{dD}{d\xi} = 2\Gamma\xi\frac{dD}{dm}\cdot\frac{dm}{dx} \tag{68}$$

a dependence on the variation of the parameter D along the body surface connected with the evolution of the external flow field properties.

This term turns out to include as well the story of the evolution of the boundary layer along the body, accounted for by the transformed variable $\xi(x)$, defined as an integral quantity (22) and can be expressed as

$$X = \left(2\Gamma\frac{dD}{dm}\right)\cdot\left(\frac{\int_0^x C\rho_e\mu_e u_e r^{2i}dx}{C\rho_e\mu_e u_e r^{2i}}\right)\frac{dm}{dx} = \phi Z$$

with $$\phi = 2\Gamma\frac{dD}{dm} = \phi(m, Pr, \sigma, \bar{t}_w, \omega) \tag{69}$$

a parameter depending only on the similar solution results and

$$Z = \xi\left(C\rho_e\mu_e u_e r^{2i}\right)^{-1}\frac{dm}{dx} \tag{70}$$

a function depending only on the known external flow field properties at the outer edge of the boundary layer.

The momentum integral equation finally assumes the form

$$q_w/\rho_e u_e H_e = \tau_w/\rho_e u_e^2\left\{D\left[1 - 2\Gamma\Sigma m\right] + \phi Z\right\} \tag{71}$$

that for true similar flows, being by definition m constant along the body surface and therefore Z=O reduces to the simple form

$$q_w/\rho_e u_e H_e = \tau_w/\rho_e u_e^2\left\{D\left[1 - 2\Gamma\Sigma m\right]\right\} \tag{72}$$

that evidentiates the effects played by the various boundary layer parameters.

THE EXTENDED REYNOLDS ANALOGY

The new form of the energy integral equation suggests by itself the idea of attempting an extension of the Reynolds analogy, that being derived from the integral boundary layer equations in the form valid in presence of velocity or pressure gradients at the edge of the boundary layer, has a wider range of applications.

Defining a modified Stanton number as

$$St^* \equiv q_w / \rho_e u_e H_e \qquad (73)$$

related to the Stanton number previously introduced (9) by the relationships

$$St = St^* \; (t_{aw} - t_w) / (T_e / T_o) \qquad (74)$$

and introducing the shear stress coefficient, related to the shear stress parameter previously defined (61),

$$C_f \equiv \tau_w / \tfrac{1}{2} \rho_e u_e^2 = 2f \qquad (75)$$

the extension of the Reynolds analogy can be easily obtained from the new form of the integral energy equation (71) written in the form:

$$St^* = 1/2 \, C_f \left\{ D \left[1 - 2\Gamma\Sigma m \right] + \Phi Z \right\} \qquad (76)$$

The modified Reynolds analogy factors S^* that is related to the classical one previously defined (12) by the sinple relation

$$S = S^* \; (t_{aw} - t_w) / (T_e / T_o) \qquad (77)$$

can be expressed as:

$$S^* = D \left[1 - 2\Gamma\Sigma m \right] + \Phi Z \qquad (78)$$

The proposed formula for the modified Reynolds analogy factory evidentaties the influence of the non similarity effects accounted for by the term.

$$\Phi Z = X = 2\Gamma\xi \, dD/d\xi = 2\Gamma \, dD/dm \cdot \xi \frac{dm}{d\xi}$$

that can be evaluated on the basis of the similar solution results factor Φ, once the external flow field properties lumped in the function Z are known.

In case of similar solutions, characterized by a fixed value of "m" along the body surface, the Reynolds analogy factor reduces to the form

$$S^* = D \left[1 - 2 \chi m \right] \qquad (79)$$

with $\chi \equiv \Gamma \Sigma$ a function, as well as D, of the velocity gradient parameter "m" that can be seen to have a strong influence on the analogy factor.

For the case of uniform external flow conditions being equal to zero the velocity gradient and the parameter "m", the Reynolds analogy factor reduces to the values of the shape parameter D, evalutated for m = o

$$S^* = (D)_{m=0} \qquad (80)$$

a value that conserves the dependence from the other parameters such as the Prandtl number, the wall temperature parameter and the parameters ω and σ previously discussed.

The knowledge of the parameters D(56), Γ (57), Σ (58), Φ(69), χ(79)evaluated once for all on the basis of known similar solutions, as functions of the velocity gradient parameter "m" with P_r, t_w, ω, and σ as parameters, allow the determination of the modified extended Reynolds analogy factor S^*.

The shear stress at the wall being obtained from the momentum integral equation it is then possible to establish the heat flux at the wall from the energy equation in a consistent way in terms of free stream parameters.

EVALUATION OF THE BOUNDARY LAYER PARAMETERS

Functional relationship for the boundary layer parameters: $D, \Gamma, \Sigma, \Phi, \chi$ appearing in the energy integral equation and therefore in the Reynolds analogy factor, and ψ, F appearing in the momentum integral equation are obtained, as several times anticipated, from similar solutions of the complete boundary layer equations.

The results of Dewey and Gross [5] are utilized to establish the relationships in terms of the five basic parameters appearing in the boundary layer equations:

m	= velocity gradient parameter	$m = \xi / u_e \, T_o/T_e \, \frac{du_e}{d\xi}$
P_r	= Prandtl number	$P_r = \mu C_p / K$
t_w	= Wall temperature parameter	$t_w = T_w / T_o$
ω	= viscosity low temperature exponent	
σ	= hypersonic parameter	$\sigma = u_e^2 / 2 H_e$

For the case of zero velocity gradient (m = O) the effects of all the other parameters is investigated to provide more generality to the results, while for the cases with positive velocity gradient (m > o) only the effect of the wall temperature parameter is investigated for a selected tern of values of the other parameters

$$P_r = 0.7 \qquad \omega = 0.7 \qquad \sigma = 1$$

A generalization of the formulas obtained, to include the effect of the variation of these three parameters kept constant, is of course possible but is beyond the present scope and will be reported in a separate note. No results for the paramter F are reported being easily obtainable from the ones relative to the parameter Σ.

All the parameters, being assumed of the form $Y = Y(m, P_r, t_w, \sigma, \omega)$ are expressed as a product of two functions

$$Y = Y_o \, Y_m$$

the first function Y_o is evaluated for a value of the velocity gradient parameter equal to zero, uniform flow condition, while the second one Y_m is defined as the value normalized with respect to the zero velocity gradient case, all other parameters being equal.

Uniform Flow Conditions

The function $Y_o = Y_o(m=0, Pr, t_w, \sigma, \omega)$ relative to the case of a velocity gradient parameter equal to zero, is in turn expressed as product of two functions:

$$Y_o = \mathcal{P} \cdot \mathcal{T}$$

the first function $\mathcal{P} = Y_o(m=0, Pr, t_w=0, \sigma, \omega)$ evaluated for a value of the wall temperature parameter equal to zero (cold wall case) evidentiates the dependence on the Prandtl number (ω, σ being kept as parameters); the second function $\mathcal{T}$ is defined as the value normalized with respect to the zero wall temperature parameter case, all toher parameters being equal, and evidentiates the influence of the wall temperature parameter (ω, σ being kept as parameters).

$$\mathcal{T} = \frac{Y_o(m=0, Pr, t_w, \sigma, \omega)}{Y_o(m=0, Pr, t_w=0, \sigma, \omega)}$$

General Case

Parameter D_o

The following functions correlating, with a maximum error of about five per cent, the similar solution results, have been obtained:

$$\mathcal{P}_D = Pr^{-a} \qquad (81)$$

$$\mathcal{T}_D = \tfrac{1}{2}\left[1 - \xi^{1/n} + (1-\xi)^n\right] \qquad (82)$$

with

$$a = 0{,}645 - 0{,}5\,\sigma \qquad (83)$$

$$n = \sigma\left[1{,}97\,Pr - (1{,}49 + 0{,}730\,\sigma)\,Pr^2\right] \qquad (84)$$

no effect of the parameter ω has been noticed.

The correlation has resulted possible only by normalizing the wall temperature with respect to the adiabatic wall temperature value, introducing the variable $\xi = t_w / t_{aw}$

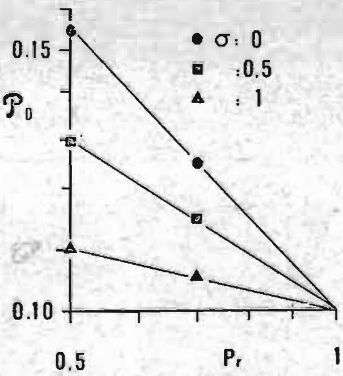
The values of the adiabatic wall temperature utilized are the ones found by Dewey and Gross by imposing $\theta'_w = 0$ and determining $T_w = T_{aw}$ by means of a double iteration procedure; these values have been correlated by the relation

$$t_{aw} = Pr^{0{,}42\,\sigma + (0{,}33 - 0{,}22\,\omega)\,\sigma^2} \qquad (85)$$

that has been found accurate within few percents.

The function $\mathcal{P}_D$ a generalization of the Colburn correction factor in the Reynolds analogy to account for the effects of the hypersonic parameter σ, is reported in figure 1, where a comparison is made with the low speed results that have allowed Nagel [20] to improve the Colburn correction factor by changing the value of "a" from 2/3 to 0,645 (value retained in the present investigation for the case $\sigma = 0$).

The correlation function for the adiabatic wall temperature is reported as a function of the Prandtl number in figure 2.



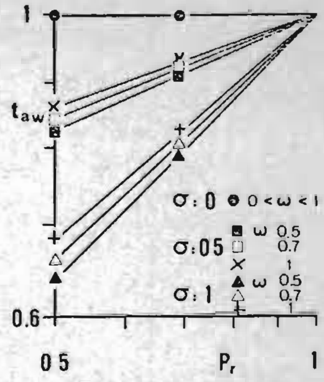


Fig. 1 (left) Parameter D_o. Function $\mathcal{P}_D(Pr, \sigma)$ variable Pr.

Fig. 2 (right) Adiabatic wall parameter $t_{aw} = T_{aw}/T_o$

The strong dependence of D from the wall temperature parameter is shown in figure 3 where the correlation function $\mathcal{T}_D$ is reported as a function of the normalized wall temperature

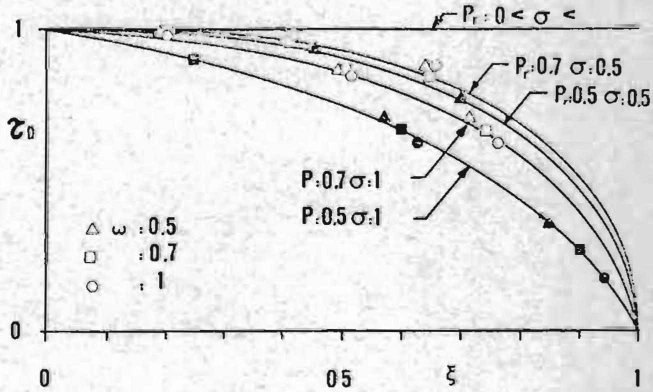


Fig. 3 Parameter D_o. Function $\mathcal{T}_D(\xi)$

Parameter Γ_o

From the investigation of the similar solution results the value of the parameter Γ_o has been found to be, for the majority of the cases, equal to one, except for few cases with $\omega = 0{,}5$ and $\sigma = 1$ for which lower values have been found; due to the small deviations from the unity value and to the fact that such results seem suspicious the simple relation

$$\Gamma_o = 1 \qquad (86)$$

has been retained.

Parameter Σ_o

The results of the similar solutions have been correlated with a maximum error of about five percent by the relationship

$$\mathcal{P}_\Sigma = b + c\,\sigma \qquad (87)$$

$$\mathcal{T}_\Sigma = 1 + d\,\xi \qquad (88)$$

with

$$b = 1{,}047 - 1{,}78\,Pr + 1{,}733\,Pr^2$$

$$c = -0{,}06 + 1{,}17\,Pr - 1{,}102\,Pr^2$$

$$d = (-4{,}6\,\sigma^2 + 13{,}9\,\sigma - 12{,}8)\,Pr^2 + (5{,}82\,\sigma^2 + -15\,\sigma + 14{,}8)\,Pr + (-1{,}22\,\sigma^2 + 1{,}13\,\sigma + 0{,}39)$$

no noticeable effect of the parameter "ω" has been found on "b" and "c", the effect on "d" is negligeable and difficult to be assessed.

The dependence on the wall temperature parameter has been again obtained by introducing the variable $\xi = t_w / t_{aw}$

The results are shown in figs. 4 and 5

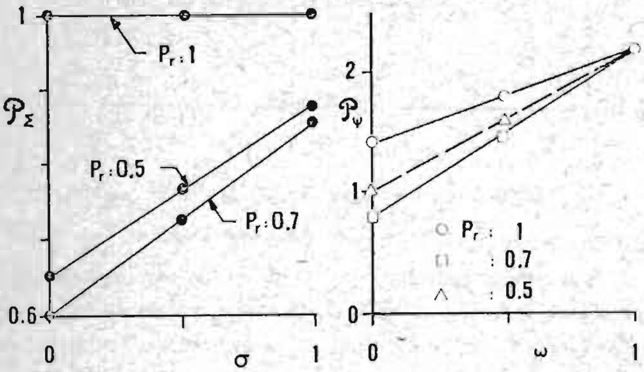


Fig. 4 (left) Parameter Σ_o. Function $\mathcal{P}_\Sigma(P_r, \sigma)$ variable σ.

Fig. 6 (right) Parameter ψ_o. Function $\mathcal{P}_\psi(P_r, \omega)$ variable ω.

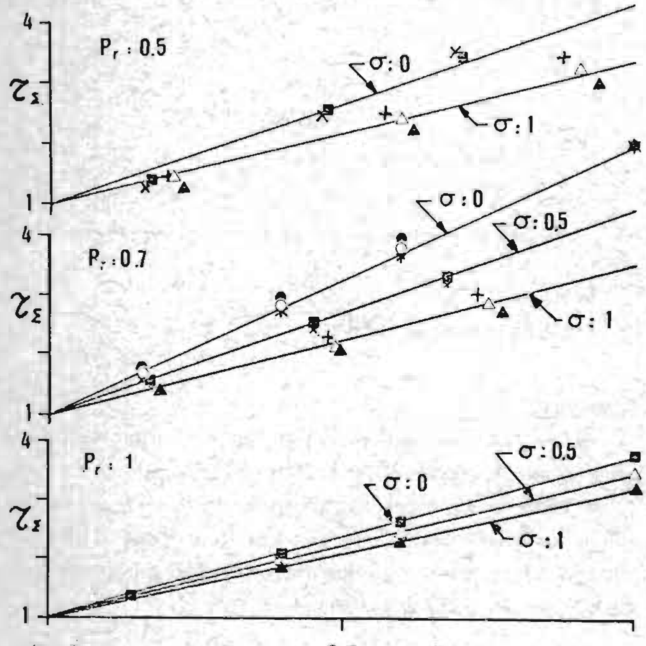


Fig. 5 Parameter Σ_o. Function $\tau_\Sigma(\xi)$. Different Prandtl values

Parameter ψ_o

From the correlation of similar solution results the relationship

$$\mathcal{P}_\psi = e + f\,\omega \qquad (89)$$

$$\tau_\psi = 1 + g\,\xi + h\,\xi^2 \qquad (90)$$

with

$$e = 1{,}2\,P_r^2 - 1{,}64\,P_r + 0{,}5$$

$$f = 1{,}2\,P_r^2 + 1{,}64\,P_r - 0{,}28$$

and g, h defined as in footnote p. 14.

have been found that approximate reasonably well the values as reported in figure 6 and 7.

For $\mathcal{P}_\psi$ no noticable dependence on σ has been found.

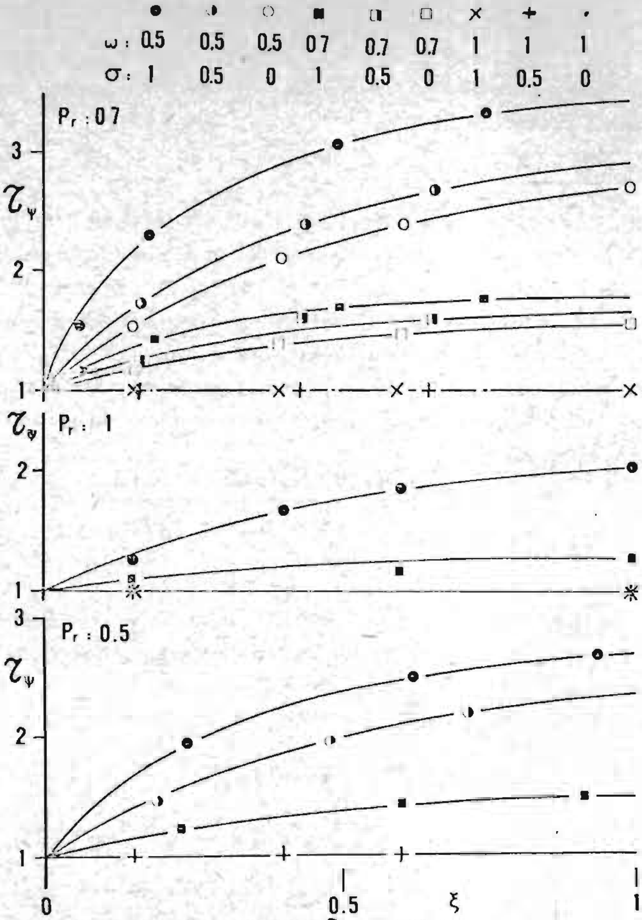


Fig. 7 Parameter ψ_o. Function $\tau_\psi(\xi)$. Different Prandtl values

Parameter χ_o

Having been found that with good accurancy $\Gamma_o = 1$ the value of χ_o is by definition equal to the value of Σ_o

Special case $P_r = 0{,}7$, $\omega = 0{,}7$, $\sigma = 1$

For the special case, typical of the supersonic flow boundary layer conditions, characterized by the tern of values above quoted for the Prandtl number P_r the viscosity parameter ω and the hypersonic parameter σ, the values of the parameters appearing in the definition of D_o, Σ_o, ψ_o previously defined are:

$a = 0{,}145$	$n = 0{,}291$	$b = 0{,}650$
$c = 0{,}219$	$d = 2{,}540$	$e = -0{,}060$
$f = 1{,}456$	$g = 1{,}417$	$h = 0{,}779$

Velocity Gradient Effects

The external flow field non uniformities, characterized by the parameter "m" defined as:

$$m = (\xi / u_e)\,(T_o / T_e)\,(d u_e / d\xi)$$

strongly influence the boundary layer parameters D, Γ, Σ, χ, ψ, Φ, F previously defined and generically called $Y = Y(m, P_r, t_w, \sigma, \omega)$; these effects have been investigated by analyzing the influence of the Prandtl number P_r, the viscosity parameter ω, and the hypersonic parameter σ, on the function Y_m defined as the value of the function Y normalized with respect to the zero velocity gradient value Y_o previously defined:

$$Y_m = y(m, Pr, t_w, \sigma, \omega) / y_o(m=0, Pr, t_w, \sigma, \omega)$$

The dependence of Y_m on such parameters has turned out to be rather complex, therefore only the results for the special case characterized by $Pr = 0,7$, $\omega = 0,7$, $\sigma = 1$ are reported and discussed in detail.

The influence of the wall temperature, that has been found to be very relevant, has been accounted for by utilizing the parameter $\xi = t_w / t_{aw}$ the results obtained from the detailed computations by Dewey and Gross [5] are correlated by functions of the form:

$$Y_m = 1 + K m^{\alpha} \qquad (91)$$

with

$$K = K(\xi ; Pr, \omega, \sigma \text{ fixed})$$

$$\alpha = \alpha(\xi ; Pr, \omega, \sigma \text{ fixed})$$

Parameter D_m

The values of the parameter D_m, see fig. 8 are found to be monotonically increasing with the value of the velocity gradient parameter "m"; the higher the value of the wall temperature the higher is the value of D_m

The correlation functions result to be :

$$K_D = 0,75 \xi + 0,434 \qquad (92)$$

$$\alpha_D = 0,091 \xi + 0,757 \qquad (93)$$

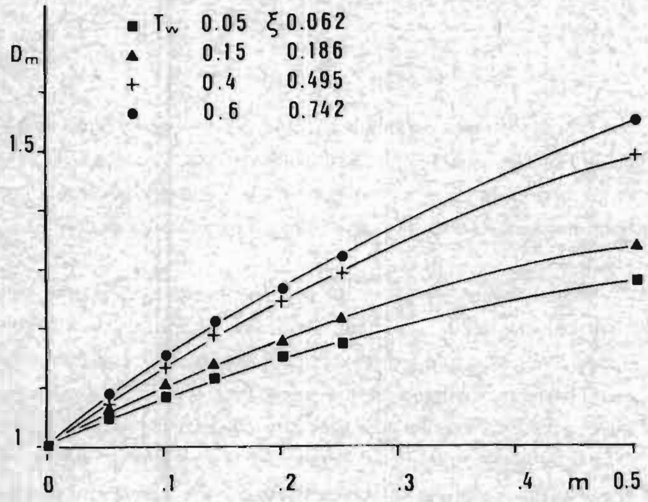


Fig. 8 Parameter $D_m(m, \xi)$ variation with m

Case $Pr = 0,7$ $\omega = 0,7$ $\sigma = 1$

Parameter Γ_m

The values of the parameter Γ_m see fig. 9, are found to be monotonically decreasing with the value of the velocity gradient parameter "m"; the higher the value of the wall temperature the lower is the value of Γ_m.

The correlation functions result to be:

$$K_\Gamma = -0,485 \xi - 0,62 \qquad (94)$$

$$\alpha_\Gamma = -0,166 \xi + 0,575 \qquad (95)$$

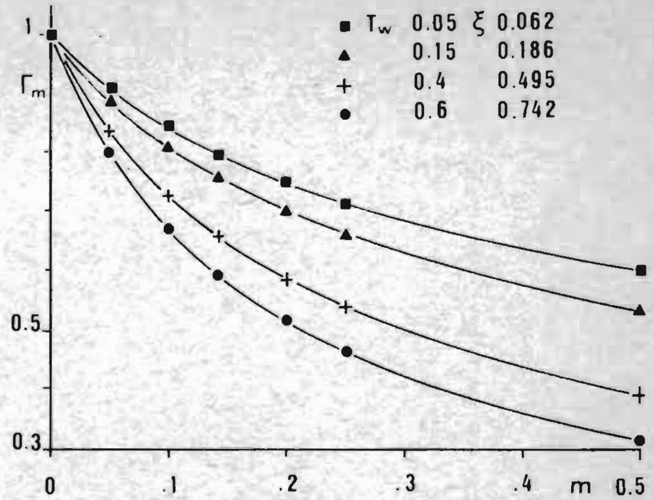


Fig. 9 Parameter $\Gamma_m(m, \xi)$ variation with m.

Case $Pr = 0,7$ $\omega = 0,7$ $\sigma = 1$

Parameter Σ_m

The values of the parameter Σ_m see fig. 10, are found to be monotonically decreasing with the value of the velocity gradient parameter "m"; the lower the value of the wall temperature the lower is the value of Σ_m.

The correlation function result to be:

$$K_\Sigma = 0,560 \xi - 0,655 \qquad (96)$$

$$\alpha_\Sigma = 0,044 \xi + 0,717 \qquad (97)$$

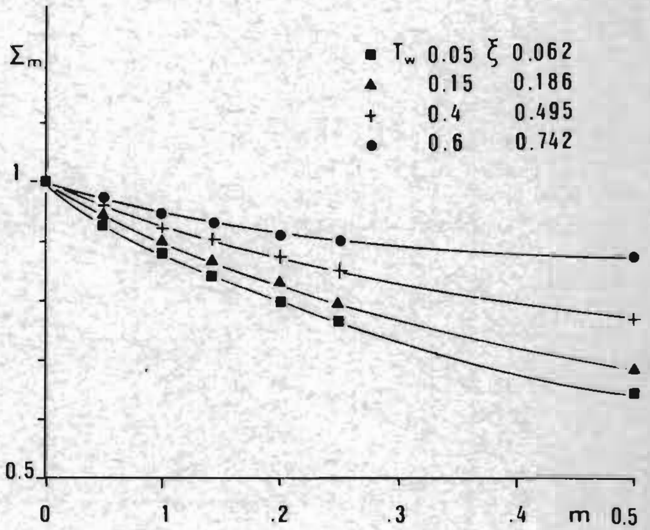


Fig. 10 Parameter $\Sigma_m(m, \xi)$ variation with m.

Case $Pr = 0,7$ $\omega = 0,7$ $\sigma = 1$

Parameter X_m

The values of the parameter X_m see fig. 11, are found to be monotonically decreasing with the value of the velocity gradient parameter "m"; the lower the value of the wall temperature the higher the value of X_m

The correlation functions result to be:

$$K_{\chi} = -0{,}127\ \xi - 0{,}913 \qquad (98)$$

$$\alpha_{\chi} = -0{,}105\ \xi + 0{,}541 \qquad (99)$$

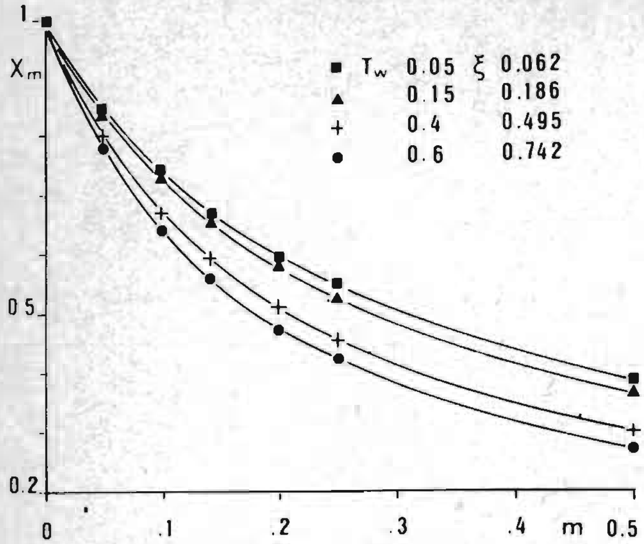


Fig. 11 Parameter $\chi_m(m,\xi)$ variation wiht m.

Parameter ψ_m

The values of the parameter ψ_m, see fig. 12, are found to be monotonically decreasing with the value of the velocity gradient parameter "m"; the higher the value of the wall temperature the higher is the value of ψ_m

The correlation functions result to be :

$$K_{\psi} = 0{,}264\ \xi + 0{,}584 \qquad (100)$$

$$\alpha_{\psi} = -0{,}286\ \xi + 0{,}833 \qquad (101)$$

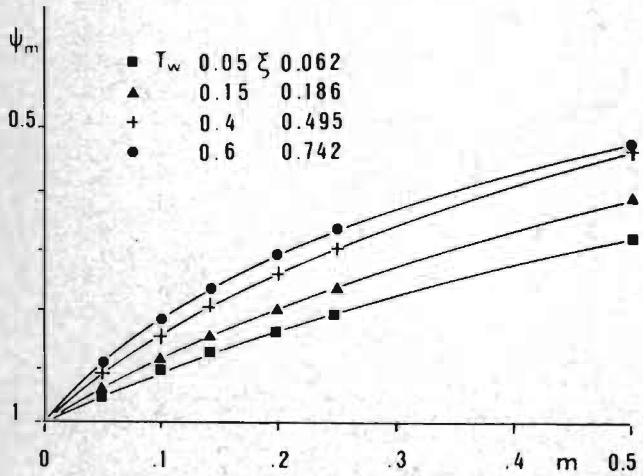


Fig. 12 Parameter $\psi_m(m,\xi)$ variation with m.

Case $Pr = 0{,}7$ $\omega = 0{,}7$ $\sigma = 1$

Non Similarity Effects

As previously discussed in the new treatment of the energy integral equation, the parameter $\Phi = 2\Gamma\ dD/dm$ allows to take into account, at least in an approximate first order effect form, the effects of the variation along the body surface of the velocity gradient parameter "m" on the new boundary layer shape parameter D, that has been introduced to consistently solve the energy integral equation.

The relationship defining the parameter Φ on the basis of the equations defining Γ and D is conducted to the form

$$\Phi = 2\,\Gamma_m\, D_o\ dD_m/dm \qquad (102)$$

In virtue of the functional correlation found for the parameter D_m the above relation assumes the form

$$\Phi = 2\,\Gamma_m\, D_o\, K_D\, \alpha_D\, m^{\alpha_D - 1} \qquad (103)$$

that allows an easy evaluation for an assigned value of the wall temperature parameter ξ. Having defined D_m only for the special case $Pr = 0{,}7$, $\omega = 0{,}7$, $\sigma = 1$ the values of Φ can be determined only for such case.

CONCLUSIONS

The consistent solution of the momentum and energy boundary layer integral equations, performed by means of the introduction of a new boundary layer shape parameter D defined as the ratio of the energy defect thickness Λ to the momentum defect thickness θ, has allowed an extension of the classical Reynolds Analogy to the more pratical cases of non uniform external flow conditions.

The Reynolds analogy factor is expressed, for the conditions of locally similar solutions validity, by the simple relation:

$$S^* = D\left[1 - 2\chi_m\right]$$

the boundary layer parameters D and χ have been related to the value of the velocity gradient parameter m characterizing the external non uniformity flow conditions, by means of correlation formulas obtained on the basis of exact results of boundary layer solutions.

The relevant effects of the basic parameters such as the wall temperature t_w, the Prandtl number Pr, the viscosity-temperature exponent ω, the hypersonic parameter σ, have been evidentiated in the correlation formulas, that have been established only for the case of zero velocity gradient $m = 0$, while for the more general case $m > 0$ only the effect of the wall temperature has been assessed having treated a special case with $Pr = 0{,}7$, $\omega = 0{,}7$, $\sigma = 1$.

The generalization of the formulas proposed to take into account also the influence of Pr, ω, σ is feasible but introduces complexity into the formalism of the correlation formulas and therefore has not been treated.

The proposed formula for the modified analogy factor evidentaties as well the influence of non similarity effects through an additional term that can be as well calculated on the basis of the similar solutions correlations.

The large deviations from the value predictable with the classical Reynolds Analogy are evidentiated, for the particular case treated, in figure 13 where the results of the correlation formulas obtained for the analogy factor S* are reported as a function of the velocity gradient parameter m .

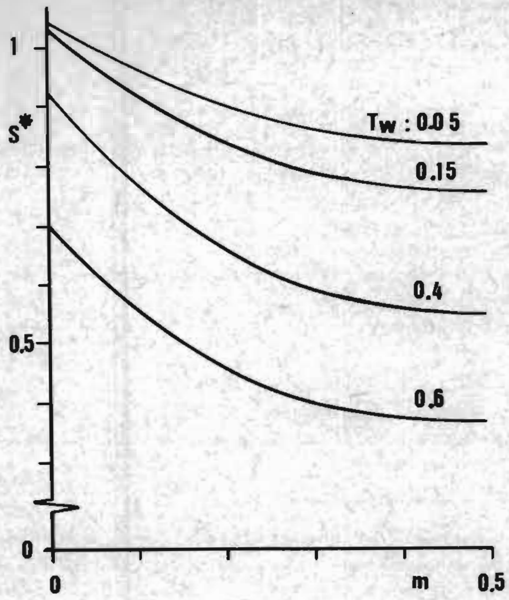


Fig. 13 Reynolds Analogy Factor S* variations with m.

The effect of the wall temperature is remarkable, the higher is the temperature the lower turns out to the value of the analogy factor S*.

The results reported clearly demonstrate the necessity of an accurate evaluation of the Reynolds Analogy factor that properly takes into account the influence of the velocity gradient parameter and of the wall temperature level; the proposed correlation formulas provide with a limited amount of computational efford and a reasonable accurancy the mean to correctly correlate the heat flux distribution along bidimensional and axisymmetric bodies to a known wall shear stress distribution.

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$$q = l\sigma + n \qquad h = p\sigma + n$$

$$l = K_1 Pr^2 + K_2 Pr + K_3 \qquad p = K_7 Pr^2 + K_8 Pr + K_9$$

$$n = K_4 Pr^2 + K_5 Pr + K_6 \qquad n = K_{10} Pr^2 + K_{11} Pr + K_{12}$$

$$K_1 = 75\,w^2 - 157,5\,w + 82,5$$
$$K_2 = -92,5\,w^2 + 202,25\,w - 109,75$$
$$K_3 = 17,5\,w^2 - 44,75\,w + 27,25$$
$$K_4 = -10,745\,w^2 + 156,4\,w - 48,99$$
$$K_5 = 21,43\,w^2 - 24,97\,w + 3,54$$
$$K_6 = -6,42\,w^2 + 9,22\,w - 2,8$$
$$K_7 = -84,17\,w^2 + 185,95\,w - 101,68$$
$$K_8 = 128,9\,w^2 - 277,53\,w + 148,6$$
$$K_9 = -15,6\,w^2 + 47,9\,w - 32,32$$
$$K_{10} = -53,17\,w^2 + 108,15\,w - 54,98$$
$$K_{11} = 88,73\,w^2 - 175,18\,w + 86,45$$
$$K_{12} = -26,58\,w^2 + 49,4\,w - 22,84$$